CONSTRUCTION OF ISOLATED LEFT ORDERINGS VIA PARTIALLY CENTRAL CYCLIC AMALGAMATION

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Abstract. We give a new method to construct isolated left orderings of groups by means of amalgamated free products, which provides a lot of new examples of isolated left orderings. We construct an isolated left ordering with various properties which previously known isolated orderings do not have.

1. Introduction

A total ordering $<_G$ on a group G is called *a left ordering* if $g <_G g'$ implies $hg <_G hg'$ for all $g, g', h \in G$. The *positive cone* of a left ordering $<_G$ is a sub-semigroup $P(<_G)$ of G defined by $P(<_G) = \{g \in G \mid 1 <_G g\}$.

The set of all left orderings of G is denoted by $\mathrm{LO}(G)$. For $g \in G$, let U_g be a subset of $\mathrm{LO}(G)$ defined by

$$U_g = \{<_G \in \mathrm{LO}(G) \mid 1 <_G g\}.$$

We equip a topology on $\mathrm{LO}(G)$ so that $\{U_g\}_{g\in G}$ is an open sub-basis of the topology. This topology is understood as follows. For a left ordering $<_G$ of G, G is decomposed as a disjoint union $G = P(<_G) \coprod \{1\} \coprod P(<_G)^{-1}$. Conversely, a sub-semigroup P of G having this property is a positive cone of a left ordering of G, so $\mathrm{LO}(G)$ is regarded as a subset of the powerset $2^{G-\{1\}}$. The topology of $\mathrm{LO}(G)$ defined as above coincides with the relative topology as the subspace of $2^{G-\{1\}}$, equipped with the powerset topology.

In this paper, we always consider *countable* groups, so we simply refer a countable group as a group unless otherwise specified. Then it is known that $\mathrm{LO}(G)$ is a compact, metrizable, and totally disconnected [9]. Moreover, $\mathrm{LO}(G)$ is either uncountable or finite [4]. Thus, as a topological space, $\mathrm{LO}(G)$ is rather similar to the Cantor set: The main difference is that the space $\mathrm{LO}(G)$ might have isolated points. Indeed, if $\mathrm{LO}(G)$ has no isolated points and is not finite, then $\mathrm{LO}(G)$ is homeomorphic to the Cantor set. We call a left ordering which is an isolated point of $\mathrm{LO}(G)$ an *isolated ordering*.

It is known that a left ordering $<_G$ is isolated if the positive cone of $<_G$ is finitely generated sub-semigroup of G. We say a finite set of non-trivial elements of G, $\mathcal{G} = \{g_1, \ldots, g_r\}$ *defines* an isolated left ordering $<_G$ of G if the positive cone of $<_G$ is generated by $\mathcal{G}$ as a semigroup. For an isolated left ordering $<_G$ of a group G, the *rank* of $<_G$ is the minimal number of generating sets of the positive cone of $<_G$ and denoted by $<_G$ by $r(<_G)$. (If $P(<_G)$ is not finitely generated semigroup we define $r(<_G) = \infty$).

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Let us call an isolated ordering $<_G$ of G is *genuine* if $\mathrm{LO}(G)$ is not a finite set so $\mathrm{LO}(G)$ contains (uncountably many) non-isolated points. In general, it is difficult to construct (genuine) isolated left orderings of groups. At this time of writing, as long as the author's knowledge, there are essentially only two families of genuine isolated left orderings.

(1) The *Dubrovina-Dubrovin ordering* $<_{DD}$ of the braid group B_n [1],[2]. Let $\sigma_1,\ldots,\sigma_{n-1}$ be the standard generator of B_n. The positive cone of the Dubrovina-Dubrovin ordering $<_{DD}$ is generated by $\{a_1,\ldots,a_{n-1}\}$, where a_i is defined by

$$a_i = (\sigma_{n-i}\sigma_{n-i+1}\cdots\sigma_{n-1})^{(-1)^i}$$

The rank of the Dubrovina-Dubrovin ordering $<_{DD}$ is $n-1$.

(2) Isolated ordering $<_A$ of the groups $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$, in other words, groups given by a presentation

$$G = \langle x, y \mid x^m = y^n \rangle.$$

The positive cone of A is generated by $\{xy^{1-n}, y\}$, and rank of $<_A$ is 2. This example is found by Navas [6] for the case $m=2$, and the author [3] for general cases.

Thus, it is desirable to find new examples or new constructions of isolated left orderings. In author's previous paper [3], we gave a general method to construct isolated orderings. We introduced a *Dehornoy-like ordering*, which is a generalization of the Dehornoy ordering of the braid groups. The Dehornoy ordering of the braid groups is the most interesting example of left orderings, and have various stimulating features and a lot of interesting interpretations. See [1] for the theory of Dehornoy ordering.

We showed that, under some mild condition which we called the Property F, Dehornoy-like orderings produces isolated orderings and vice versa. Indeed, the above two families of isolated orderings are derived from Dehornoy-like orderings. However, it is difficult to find new examples of Dehornoy-like orderings since the definition of Dehornoy-like ordering includes rather mysterious and complicated combinatorics.

The aim of this paper is to give a new construction of isolated left orderings by means of *partially central cyclic amalgamation*. From two groups having (not necessarily genuine) isolated orderings, we construct a new group having an isolated left ordering. In almost all cases, the constructed isolated orderings are genuine. Our construction can be seen as an extension of (2) of the known examples, but is not related to Dehornoy-like ordering construction.

The following is the main result of this paper. Recall that for $g \in G$ and a left ordering $<_G$ of G, $<_G$ called a *g-right invariant* if $<_G$ is preserved by the right multiplication of g, that is, $a <_G b$ implies $ag <_G bg$ for all $a, b \in G$.

Theorem 1 (Construction of isolated left ordering via partially central cyclic amalgamation)**.** *Let G and H be finitely generated groups. Let z_G be a non-trivial central element of G, and z_H be non-trivial element of H.*

Let $\mathcal{G} = \{g_1,\ldots,g_m\}$ be a finite generating set of G which defines an isolated left ordering $<_G$ of G. We take a numbering of elements of $\mathcal{G}$ so that $1 <_G g_1 <_G \cdots <_G g_m$ holds. Similarly, let $\mathcal{H} = \{h_1,\ldots,h_n\}$ be a finite generating set of H which defines an isolated left ordering $<_H$ of H such that the inequality $1 <_H h_1 <_H$

$\ldots <_H h_n$ holds. We assume the cofinality assumption **CF(G)** *and* **CF(H)**, *and the invariance assumption* **INV(H)**.

CF(G): *$g_i <_G z_G$ hold for all i.*
CF(H): *$h_i <_H z_H$ hold for all i.*
INV(H): *$<_H$ is z_H-right invariant.*

Let $X = G *_{\mathbb{Z}} H = G *_{\langle z_G = z_H \rangle} H$ be an amalgamated free product of G and H. For $i = 1, \ldots, m$, let $x_i = g_i z_H^{-1} h_1$. Then we have the following results:

(1) *The generating set $\{x_1, \ldots, x_m, h_1, \ldots, h_n\}$ of X defines an isolated left ordering $<_X$ of X.*
(2) *The isolated ordering $<_X$ does not depend on a choice of a generating sets $\mathcal{G}$ and $\mathcal{H}$. Thus, $<_X$ only depends on isolated orderings $<_G, <_H$ and z_G, z_H.*
(3) *The natural inclusions $\iota_G : G \to X$ and $\iota_H : H \to X$ are order-preserving homomorphism.*
(4) *$1 <_X x_1 <_X \cdots <_X x_m <_X h_1 <_X \cdots <_X h_n <_X z_H = z_G$. Moreover, $z = z_G = z_H$ is $<_X$-positive cofinal and the isolated ordering $<_X$ is z-right invariant.*
(5) *$r(<_X) \leq r(<_G) + r(<_H)$.*
(6) *Let Y be a non-trivial proper subgroup of X. If Y is $<_X$-convex, then $Y = \langle x_1 \rangle$.*

We call the construction of isolated ordering described in Theorem 1 the *partially central cyclic amalgamation construction*.

As we will see in Lemma 3 in Section 2.1, the cofinality assumption **CF(G)** (resp. **CF(H)**) are understood as an assumption on z_G and $<_G$ (resp. z_H and $<_H$) and Theorem 1 (2) shows that the choice of generating sets $\mathcal{G}$ and $\mathcal{H}$ is not important, though it is useful to describe and understand the isolated ordering $<_X$. Thus, the generating sets $\mathcal{G}$ and $\mathcal{H}$ play rather auxiliary roles and are not essential in our partially central cyclic amalgamation construction.

Theorem 1 (3) shows that the partially central cyclic amalgamation construction can be seen as a mixing of two isolated orderings $<_G$ and $<_H$. We remark that Theorem 1 (4) implies that we can iterate the partially central cyclic amalgamation construction. Thus, we can actually produce many isolated orderings by using the partially central cyclic amalgamation construction.

Unlike the partial central cyclic amalgamation (the amalgamated free product in Theorem 1), the usual free product does not preserve the property that the group has an isolated left orderings. For example, the free group of rank two $F_2 = \mathbb{Z} * \mathbb{Z}$ has no isolated ordering [5], whereas the infinite cyclic group $\mathbb{Z}$ has isolated orderings, since it admits only two left orderings. Indeed, recently Rivas [7] proved that the free product of groups do not have any isolated left orderings.

Moreover, the proof of Theorem 1 (1) given in this paper is constructive, so the proof actually provide an algorithm to compute the isolated ordering $<_X$. In particular, the isolated ordering $<_X$ can be determined algorithmically if we have algorithms to compute isolated orderings $<_G$ and $<_H$. See Section 2.7.

The plan of this paper is as follows: In Section 2 we give a proof of Theorem 1. The main technical tool in th proof of Theorem 1 is a *reduced standard factorization*, which serves as a some kind of normal form of elements in X. In Section 3 we give some examples and observe new features of isolated orderings. We will pose

various conjectures on the structure of groups having isolated orderings.

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2. Construction of isolated left orderings

Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a finite generating set of G and let $\mathcal{S}^{-1} = \{s_1^{-1}, \ldots, s_n^{-1}\}$. We denote by $\mathcal{S}^*$ the free semigroup generated by $\mathcal{S}$. That is, $\mathcal{S}^*$ is a set of non-empty words on $\mathcal{S}$. We say an element of $\mathcal{S}^*$ (resp. $(\mathcal{S}^{-1})^*$) an *$\mathcal{S}$-positive word* (resp. an $\mathcal{S}$-negative word). We will often use a symbol $P(\mathcal{S})$ (resp. $N(\mathcal{S})$) to represent an $\mathcal{S}$-positive (resp. $\mathcal{S}$-negative) word.

2.1. **Cofinality and Invariance assumption.** First of all we review the assumptions in the statement of Theorem 1 again, and deduce their consequences. This clarifies the role of each hypothesis in Theorem 1.

Let G and H be countable groups having an isolated left ordering $<_G$, $<_H$ respectively. Let $z_G \in G$ be a non-trivial central element of G, and let z_H be a non-trivial element of H, which might be not central. We consider the group X obtained as an amalgamated free product

$$X = G *_{\mathbb{Z}} H = G *_{\langle z_G = z_H \rangle} H,$$

which is again countable.

Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a generating set of G which defines an isolated left ordering $<_G$ of G. We take a numbering of elements of $\mathcal{G}$ so that $1 <_G g_1 <_G \cdots <_G g_m$ holds. Similarly, let $\mathcal{H} = \{h_1, \ldots, h_n\}$ be a generating set of H which defines an isolated left ordering $<_H$ of H, and we assume that the inequality $1 <_H h_1 <_H \cdots <_H h_n$ holds.

Recall that an element $g \in G$ is called the *$<_G$-minimal positive element* if g is the $<_G$-minimal element among the positive cone $P(<_G)$. In other words, the inequality $1 <_G g' \leq_G g$ implies $g = g'$. A left ordering $<_G$ is called *discrete* if $<_G$ has the $<_G$-minimal positive element. Otherwise, $<_G$ is called *dense*. As next lemma shows, the assumptions on the numbering of $\mathcal{G}$ implies that g_1 (resp. h_1) is the $<_G$-minimal (resp. $<_H$-minimal) positive element, hence g_1 (resp. h_1) is independent of a choice of the generating set $\mathcal{G}$ (resp. $\mathcal{H}$).

Lemma 1. *Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a generating set of a group G which defines an isolated left ordering $<_G$ of G. Assume that g_1 is the $<_G$-minimal element of the set $\mathcal{G}$. Then g_1 is the $<_G$-minimal positive element. In particular, $<_G$ is discrete. Moreover, $<_G$ is g_1-right invariant.*

Proof. Assume $g \in G$ satisfies the inequality $1 <_G g \leq_G g_1$. Since $1 <_G g$, g_1 is written as a $\mathcal{G}$-positive word $g = g_{i_1} \cdots g_{i_l}$. If $i_1 \neq 1$, then $g_1^{-1} g_{i_1} >_G 1$, which is a contradiction. If $l > 1$, then $g_1^{-1} g_{i_1} >_G 1$ unless $i_1 = 1$. So we conclude $g = g_1$.

The g_1-right invariance of the ordering $<_G$ now follows from the fact g_1 is $<_G$-minimal positive element: If $a >_G b$, then $b^{-1} a g_1 >_G b^{-1} a >_G 1$. Thus, $b^{-1} a g_1 >_G g_1$ so $a g_1 >_G b g_1$. ∎

To obtain an isolated ordering of X from $<_G$ and $<_H$, we impose the following assumptions, which we call the *cofinality assumption* and the *invariance assumption*.

CF(G): $g_i <_G z_G$ hold for all i.
CF(H): $h_i <_H z_H$ hold for all i.
INV(H): $<_H$ is z_H-right invariant.

We here remark that the invariance assumption for $<_G$ is automatically satisfied: that is, $<_G$ is z_G-right invariant since we have chosen z_G as a central element.

First we observe the following simple lemma.

Lemma 2. *Let $<_H$ be a discrete left ordering of a group H, and let h_1 be the $<_H$-minimal positive element, If $<_H$ is an h-right invariant for $h \in H$, then h commutes with h_1.*

Proof. $<_H$ is an h-right invariant, so $hh_1h^{-1} >_H 1$ and $h^{-1}h_1h >_H 1$. h_1 is the $<_H$-minimal positive element, $hh_1h^{-1} \geq_H h_1$ and $h^{-1}h_1h \geq_H h_1$. Thus, we get $hh_1 \geq_H h_1h$ and $h_1h \geq_H hh_1$, hence $hh_1 = h_1h$. □

By Lemma 1 and Lemma 2, the invariance assumption [**INV(H)**] implies that z_H commutes with h_1.

Recall that an element $g \in G$ and a left-ordering $<_G$ of G, g is called $<_G$-*cofinal* if for all $g' \in G$, there exists integers m and M such that $g^m <_G g' <_G g^M$ holds. Although the cofinality assumptions [**CF(G)**] and [**CF(G)**] involves the generating sets $\mathcal{G}$ and $\mathcal{H}$, these assumptions should be regarded as assumptions on z_G, z_H and isolated orderings $<_G$, $<_H$ as the next lemma shows.

Lemma 3. *Assume the invariance assumption* [**INV(H)**] *is satisfied. A generating set $\mathcal{H}$ satisfying the cofinality assumption* [**CF(H)**] *exists if and only if z_H is $<_H$-positive cofinal and $H \neq \langle z_H \rangle$. Moreover, in such case we may choose a generating set $\mathcal{H}$ so that the cardinal of $\mathcal{H}$ is equal to the rank of the isolated ordering $<_H$.*

Proof. It is easy to see if a generating set $\mathcal{H}$ satisfies the cofinality assumption [**CF(H)**], then z_H is $<_H$-positive cofinal and $H \neq \langle z_H \rangle$. We show that under the invariance assumption [**INV(H)**], if z_H is $<_H$-positive cofinal and $H \neq \langle z_H \rangle$, then we can choose a generating set $\mathcal{H} = \{h_1, \ldots, h_k\}$ which defines the ordering $<_H$ so that $\mathcal{H}$ satisfies [**CF(H)**] and $k = r(<_H)$.

Let us take a generating set $\mathcal{H}' = \{h'_1, \ldots, h'_k\}$ of H which defines the isolated ordering $<_H$ and $k = r(<_H)$. Assume that $h'_1 <_H \ldots <_H h'_s \leq_H z_H <_H h'_{s+1} <_H \cdots <_H h'_k$. Since z_H is $<_H$-cofinal, for each i there is a non-negative integer N_i such that $1 <_H z_H^{-N_i} h'_i \leq z_H$. Let $h_i = z_H^{-N_i} h'_i$. By assumption, $h'_i = h_i$ if $i \leq s$.

Since $H \neq \langle z_H \rangle$, $z_H >_H h'_1 = h_1$. So if necessary, by replacing h_i with ${h_1}^{-1}h_i$, we can assume that $h_i \neq z_H$ for all i.

We show that z_H is written by a $\{h'_1, \ldots, h'_s\}$-positive word. Assume that $z_H = Vh'_iW$, where $i > s$ and V, W are $\mathcal{H}'$-positive or non-empty words. Then $z_HW^{-1} = Vh'_iWW^{-1} = Vh'_i >_H Vz_H$, hence we get $1 \geq_H W^{-1} >_H z_H^{-1}Vz_H$. However, $<_H$ is z_H-right invariant, $z_H^{-1}Vz_H \geq_H 1$. This is a contradiction.

Therefore the generating set $\mathcal{H} = \{h_1, \ldots, h_k\}$ also defines the isolated ordering $<_H$. So $\mathcal{H}$ is a generating set which satisfies the cofinality assumption [**CF(H)**] with cardinal $k = r(<_G)$. □

Thus, we can always find a generating set $\mathcal{H}$ which defines $<_H$ and satisfies the cofinality assumption [**CF(H)**] if the condition on $<_H$ and z_H in Lemma 3 is satisfied. Moreover, if necessary we may choose $\mathcal{H}$ so that the cardinal of $\mathcal{H}$ is equal to the rank of $<_H$. Since for z_G and $<_G$ the invariance assumptions is automatically

satisfied, we conclude that we can always find a generating set $\mathcal{G}$ which defines $<_G$ and satisfies the cofinality assumption $[\mathbf{CF(G)}]$ if z_G is $<_G$-positive cofinal and $G \neq \langle z_G \rangle$.

Now we put $\Delta_H = z_H h_1^{-1}$. Since z_H and h_1 do not depend on a choice of a generating set $\mathcal{H}$, so is Δ_H. As an element of H, Δ_H is characterized by the following property.

Lemma 4. *Δ_H is the $<_H$-maximal element which is strictly smaller than z_H.*

Proof. Assume that $z_H h_1^{-1} = \Delta_H \leq_H h <_H z_H$ holds for some $h \in H$. Then $h_1^{-1} \leq_H z_H^{-1} h <_H 1$. By Lemma 1, h_1^{-1} is the $<_H$-maximal element which is strictly smaller than 1, we get $z_H^{-1} h = h_1^{-1}$, hence $h = z_H h_1^{-1}$. □

Finally, we put $x_i = g_i \Delta_H^{-1} = g_i z_H^{-1} h_1$ and let $\mathcal{X} = \{x_1, \ldots, x_m\}$. Then $\{\mathcal{X}, \mathcal{H}\}$ generates the group X. The following lemma is rather obvious, but plays an important role in the proof of Theorem 1.

Lemma 5. *$z_H = z_G$ commutes with all x_i.*

Proof. By Lemma 2, z_H commutes with $\Delta_H = z_H h_1^{-1}$. Since $z_H = z_G$ commutes with all g_i, we conclude that z_H commutes with all $x_i = g_i \Delta_H^{-1}$. □

2.2. **Property A and Property C criterion.** To prove that $\{\mathcal{X}, \mathcal{H}\}$ defines an isolated left ordering $<_X$ of X, we use the following criterion which was used in studying the Dehornoy ordering of the braid groups [1], or Dehornoy-like orderings [3]. Here we give the most general form of this kind of argument.

Definition 1. Let $\mathcal{S} = \{s_1, \ldots, s_m\}$ be a generating set of a group G and let W be a sub-semigroup of $(\mathcal{S} \cup \mathcal{S}^{-1})^*$.

1. We say W has the *Property A (Acyclic Property)* if no words in W represent the trivial element of G.
2. We say W has the *Property C (Comparison Property)* if for each non-trivial element $g \in G$, either g or g^{-1} is represented by a word $w \in W$.

Proposition 1. *Let W be a sub-semigroup of $(\mathcal{S} \cup \mathcal{S}^{-1})^*$. Let $P = \pi(W)$, where $\pi : (\mathcal{S} \cup \mathcal{S}^{-1})^* \to G$ be the natural projection. Then P is equal to a positive cone of a left ordering of G if and only if W has Property A and C.*

Proof. If W is a positive cone of a left ordering, then it is obvious that W has Property A and C. We show the converse. Since W is a sub-semigroup, P is a sub-semigroup of G. By Property C, $G = P \cup \{1\} \cup P^{-1}$. Property A implies $1 \notin P$, hence G is decomposed as a disjoint union, $G = P \coprod \{1\} \coprod P^{-1}$. This shows that P is a positive cone of a left ordering. □

Definition 2. The set of words W in Proposition 1 is called a *language defining a left-ordering $<_G$*.

It is an interesting problem to ask whether or not one can choose a language defining an arbitrary left-ordering $<_G$ so that it is a regular language over finite alphabets: This is related to order-decision problem which we will consider in Section 2.7, but in this paper we will not treat this problem.

As a special case, we get a criterion for a finite generating set to define an isolated ordering, which will be used to show $\{\mathcal{X}, \mathcal{H}\}$ indeed defines an isolated ordering.

Corollary 1. *A finite generating set $\mathcal{G} = \{g_1, \ldots, g_m\}$ of a group G defines an isolated ordering of G if and only if the following condition* [**Property A**] *and* [**Property C**] *hold:*

Property A: *If $g \in G$ is represented by a $\mathcal{G}$-positive word, then $g \neq 1$.*
Property C: *If $g \neq 1$, then g is represented by either an $\mathcal{G}$-positive or an $\mathcal{G}$-negative word.*

2.3. **Reduced standard factorization.** Now we begin to show that $\{\mathcal{X}, \mathcal{H}\}$ indeed defines an isolated left ordering of X. As the first step of the proof, we introduce a notion of standard factorization.

Let PX be a sub-semigroup of X generated by $\mathcal{X} = \{x_1, \ldots, x_m\}$. A standard factorization of $x \in X$ is a factorization of $x \in X$ of the form

$$\mathcal{F}(x) = rp_1q_1 \cdots p_lq_l$$

where $r, q_1, \cdots, q_l \in H$, $p_1, \ldots, p_l \in PX$ which satisfies the conditions

(1) $q_i >_H 1$ $(i \neq l)$, and $q_l \geq_H 1$.
(2) $q_i \neq z_H^N$ for all $N > 0$.

For a standard factorization $\mathcal{F}(x) = rp_1q_1 \cdots p_lq_l$, we say l is a *complexity* of a standard factorization $\mathcal{F}(x)$, and denote by $c(\mathcal{F})$.

A *distinguished subfactorization* of a standard factorization $\mathcal{F}(x)$ is a part of the standard factorization $\mathcal{F}(x)$ of the form

$$w = x_a q_i p_{i+1} q_{i+1} \cdots p_{i+r} q_{i+r}.$$

which satisfies the two conditions

(1) $q_j = \Delta_H$ for all $j = i, i+1, \ldots, i+r$.
(2) $p_j \in \{x_1, \ldots, x_m\}$ for all $j = i+1, \ldots, i+r$.
(3) $p'_i = p_i x_a^{-1} \in PX \cup \{1\}$.

Since $g_i = x_i \Delta_H$, a distinguished sub-factorization w is naturally regarded as a $\mathcal{G}$-positive word. We will often denote a distinguished subfactorization w by $[g]$, by using an element $g \in G$ represented by a $\mathcal{G}$-positive word w, as

$$\begin{aligned} x &= rp_1q_1 \cdots p_{i-1}q_{i-1}p'_{i+1}(x_a q_i \cdots p_{i+r}q_{i+r})p_{i+r+1}q_{i+r+1} \cdots p_lq_l \\ &= rp_1q_1 \cdots p_{i-1}q_{i-1}p'_{i+1}[g]p_{i+r+1}q_{i+r+1} \cdots p_lq_l. \end{aligned}$$

Let us take x_u so that $p'_{i+r+1} = x_u^{-1} p_{i+r+1} \in PX \cup \{1\}$. We say a distinguished subfactorization w is *reducible* if $gg_u \geq_G z_G$ holds for some choice of such x_u. We say a distinguished subfactorization w is *maximal* if there are no other distinguished subfactorization which contains w. That is, $q_{i-1} \neq \Delta_H$, and $q_{i+r+1} \neq \Delta_H$ if $p_{i+r+1} \in \mathcal{X} = \{x_1, \ldots, x_m\}$.

Now we define the notion of a *reduced standard factorization*, which plays an important role in the proof of both Property A and Property C.

Definition 3 (Reduced standard factorization)**.** Let $\mathcal{F}(x) = rp_1q_1 \cdots p_lq_l$ be a standard factorization. We say $\mathcal{F}$ is *reduced* if $q_i <_H z_H$ for all i and $\mathcal{F}$ contains no reducible distinguished subfactorization.

First we show the existence of the reduced standard subfactorization. The proof of next lemma utilize the standard form of amalgamated free product, and mainly works in the generating set $\{\mathcal{G}, \mathcal{H}\}$.

Lemma 6. *Every element $x \in X$ admits a reduced standard subfactorization.*

Proof. Since X is an amalgamated free product of G and H, every $x \in X$ is written as

$$x = q_0 w_1 q_1 w_2 q_2 \cdots w_k q_k$$

where $q_i \in H$, $w_i \in G$, and $q_i \neq z_H^N$ and $w_i \neq z_G^N$ for any $N \in \mathbb{Z}$ and $i > 0$.

Since z_G is $<_G$-cofinal, for each $i > 0$ there exists N_i which satisfies

$$z_G^{N_i} <_G w_i <_G z_G^{N_i+1}.$$

We put $w_i^* = z_G^{-N_i} w_i$. Then w_i^* satisfies the inequality

$$1 <_G w_i^* <_G z_H$$

Similarly, z_H is $<_H$-cofinal, for each $i > 0$, there exists M_i which satisfies the inequality

$$z_H^{M_i} \leq_H \Delta_H q_i <_H z_H^{M_i+1}.$$

Let $L_i = \sum_{j>i}(N_j + M_j)$, and put $q_i^* = z_H^{-L_i}(z_H^{-M_i}\Delta_H q_i) z_H^{L_i}$. Since $<_H$ is z_H-right invariant, $1 \leq_H q_i^* <_H z_H$ holds. Since we have assumed that $q_i \neq z_H^N$, $q_i^* \neq \Delta_H$. Thus, $1 \leq_H q_i^* <_H \Delta_H$.

Then we get a reduced standard factorization of x as follows.

$$\begin{aligned}
x &= q_0 w_1 q_1 \cdots w_l q_l \\
&= q_0 (z_G^{N_1} w_1^*) q_1 (z_G^{N_2} w_2^*) \cdots (z_G^{N_{l-1}} w_{l-1}^*) q_{l-1} (z_H^{N_l} w_l^*) q_l \\
&= (q_0 z_H^{N_1}) w_1^* (q_1 z_H^{N_2}) \cdots w_{l-1}^* (q_{l-1} z_H^{N_l}) w_l^* q_l \\
&= (q_0 z_H^{N_1}) w_1^* (q_1 z_H^{N_2}) \cdots w_{l-1}^* (q_{l-1} z_H^{N_l}) w_l^* \Delta_H^{-1} z_H^{M_l} (z_H^{-M_l} \Delta_H q_l) \\
&= (q_0 z_H^{N_1}) w_1^* (q_1 z_H^{N_2}) \cdots w_{l-1}^* \Delta_H^{-1} z_H^{N_l+M_l} (z_H^{-N_l-M_l} \Delta_H q_{l-1} z_H^{N_l+M_l}) (w_l^* \Delta_H^{-1}) q_l^* \\
&= (q_0 z_H^{N_1}) w_1^* (q_1 z_H^{N_2}) \cdots z_H^{N_l+M_l} (w_{l-1}^* \Delta_H^{-1}) q_{l-1}^* (w_l^* \Delta_H^{-1}) q_l^* \\
&= \cdots \\
&= (q_0 z_H^{L_0}) (w_1^* \Delta_H^{-1}) q_1^* \cdots (w_{l-1}^* \Delta_H^{-1}) q_{l-1}^* (w_l^* \Delta_H^{-1}) q_l^*.
\end{aligned}$$

Now we choose an arbitrary $\mathcal{G}$-positive word expression of each w_i, and rewrite them by the generators $\{\mathcal{X}, \mathcal{H}\}$ by using the relation $g_i = x_i \Delta_H$. Then we get a standard factorization $\mathcal{F}(x)$. By construction, all distinguished sub-factorization is derived from w_i^*, so all distinguished sub-factorization are not reducible.

□

2.4. **Reducing operation and the proof of Property A.** In the proof of Lemma 6 given in previous section, we mainly use the generating set $\{\mathcal{G}, \mathcal{H}\}$. In this section we give an alternative way to get a reduced standard factorization, which works mainly in the generating set $\{\mathcal{X}, \mathcal{H}\}$.

For a standard factorization $\mathcal{F}(x)$, let $d(\mathcal{F})$ be the number of maximal reducible distinguished subfactorizations. We may regard a pair of non-negative integers $(c(\mathcal{F}), d(\mathcal{F}))$ as a complexity of standard factorization. We say a standard factorization $\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l$ is *pre-reduced* if $1 <_H q_i <_H z_H$ holds for all i.

The following Lemma 7 and Lemma 8 gives another proof of existence of a reduced standard factorization.

Lemma 7 (Existence of pre-reduced standard factorization)**.** *Every element $x \in X$ admits a pre-reduced standard factorization.*

Proof. Since $x_i^{-1} = \Delta_H g_i^{-1} = \Delta_H z_H^{-1}(z_G g_i^{-1})$, every element x has a standard factorization

$$\mathcal{F}(x) = rp_1q_1\cdots p_lq_l.$$

For each i, take $M_i \geq 0$ so that $z_H^{M_i} \leq_H q_i <_H z_H^{M_i+1}$. Let $L_i = \sum_{j\geq i} M_i$ and $q_i^* = z_H^{-L_i} q_i z_H^{L_{i+1}} = z_H^{-L_{i+1}}(z_H^{-M_i} q_i) z_H^{L_{i+1}}$. Since $<_H$ is z_H-right invariant, $1 \leq_H q_i^* <_H z^H$. Therefore, we get a pre-reduced standard factorization

$$\begin{aligned}
x &= rp_1q_1\cdots p_lq_l \\
&= rp_1q_1\cdots p_{l-1}q_{l-1}p_l(z_H^{M_l}q_l^*) \\
&= rp_1q_1\cdots p_{l-1}(q_{l-1}z_H^{M_l})p_lq_l^* \\
&= rp_1q_1\cdots p_{l-1}z_H^{-M_l-M_{l-1}}(z_H^{-M_l-M_{l-1}}q_{l-1}z_H^{M_l})p_lq_l^* \\
&= rp_1q_1\cdots p_{l-1}z_H^{-L_{l-1}}q_{l-1}^*p_lq_l^* \\
&= \cdots \\
&= (rz_H^{-L_0})p_1q_1^*\cdots p_lq_l^*.
\end{aligned}$$

□

Lemma 8 (Reducing operation)**.** *Let $\mathcal{F}(x) = rp_1q_1\cdots p_lq_l$ is pre-reduced standard factorization. If $\mathcal{F}(x)$ contains a reducible distinguished subfactorization, then we can find another pre-reduced standard factorization $\mathcal{F}'(x) = r'p_1'q_1'\cdots$ which satisfies either $c(\mathcal{F}') < c(\mathcal{F})$ or $d(\mathcal{F}') < d(\mathcal{F})$. Moreover, if $r >_H 1$ then $r' >_H 1$.*

Proof. Let g be a maximal reducible distinguished subfactorization. Then the pre-reduced standard factorization $\mathcal{F}(x)$ is written as

$$\mathcal{F}(x) = rp_1q_1\cdots p_{i-1}q_{i-1}p_i'[g]x_up_s'q_s\cdots p_lq_l$$

Now take $N > 0$ so that $z_G^N <_G gg_u \leq_G z_G^{N+1}$.

$$\begin{aligned}
x &= rp_1q_1\cdots p_{i-1}q_{i-1}p_i'[g]x_up_s'q_s\cdots p_lq_l \\
&= rp_1q_1\cdots p_{i-1}q_{i-1}p_i'z_G^N(z_G^{-N}gg_u)\Delta_H^{-1}p_s'q_s\cdots p_lq_l
\end{aligned}$$

For $j < i$, let $p_j^* = z_H^{-N}p_jz_H^N$ and let $p_i^* = z_H^{-N}p_i'z_H^{-N}$. Then,

$$x = (rz_H^N)p_1^*q_1\cdots p_{i-1}^*q_{i-1}p_i^*(gg_uz_G^{-N})\Delta_H^{-1}p_s'q_s\cdots p_lq_l$$

First of all, assume that $(z_G^{-N}gg_u) = z_G = z_H$. Then

$$\begin{aligned}
x &= (rz_H^N)p_1^*q_1\cdots p_{i-1}^*q_{i-1}(p_i^*z_H\Delta_H^{-1}p_s')q_s\cdots p_lq_l \\
&= (rz_H^N)p_1^*q_1\cdots p_{i-1}^*q_{i-1}(p_i^*h_1p_s')q_s\cdots p_lq_l
\end{aligned}$$

In this case, by modifying the last standard factorization into a pre-reduced standard factorization as in the proof of Lemma 7, we get a new pre-reduced standard factorization $\mathcal{F}'$ of x which satisfies $c(\mathcal{F}') < c(\mathcal{F})$.

Next assume that $(gg_uz_G^{-N}) \neq z_G$. Then let $g' = (z_G^{-N}gg_u)g_1^{-1}$. Then $1 \leq_G g' < z_Gg_1^{-1}$, and we get a new pre-reduced standard factorization

$$\begin{aligned}
\mathcal{F}'(x) &= (rz_H^N)p_1^*q_1\cdots p_{i-1}^*q_{i-1}p_i^*g'g_1\Delta_H^{-1}p_s'q_s\cdots p_lq_l \\
&= (rz_H^N)p_1^*q_1\cdots p_{i-1}^*q_{i-1}p_i^*g'(x_1p_s')q_s\cdots p_lq_l.
\end{aligned}$$

$p_i^* = \Delta_H$ if and only if $p_i = \Delta_H$, hence $d(\mathcal{F}') < d(\mathcal{F})$. □

Now we are ready to prove Property A.

Proposition 2 (Property A). *If x is expressed as an $\{\mathcal{X},\mathcal{H}\}$-positive word, then $x \neq 1$.*

Proof. Assume that x is expressed by an $\{\mathcal{X},\mathcal{H}\}$-positive word. Such a word expression naturally regarded as a standard factorization. By the proof of Lemma 7, we can modify such a standard factorization so that it is pre-reduced, preserving the property that it is also an $\{\mathcal{X},\mathcal{H}\}$-positive word. By Lemma 8, we may modify the $\{\mathcal{X},\mathcal{H}\}$-positive pre-reduced standard expression $\mathcal{F}(x)$ so that it is $\{\mathcal{X},\mathcal{H}\}$-positive reduced standard factorization.

Now let us rewrite $\mathcal{F}(x)$ as a word on $\{\mathcal{G},\mathcal{H}\}$ as follows. First we replace each distinguished subword $[g]$ in $\mathcal{F}(x)$ by the corresponding $\mathcal{G}$-positive word. Then we remove the rest of x_i by using the relation $x_i = g_i\Delta_H^{-1}$. Thus, we write $x = W_0V_1W_1\cdots V_nW_n$, where W_i is a word on $\mathcal{H}^{\pm 1}$ and V_i is a word on $\mathcal{G}^{\pm 1}$. Since $\mathcal{F}(x)$ is a reduced standard expression, $V_i, W_i \notin \langle z_H\rangle$ for $i > 0$. This implies that $x \neq 1$, since $X = G *_{\langle z_G = z_H\rangle} H$.

□

2.5. **Proof of Property C.** Next we give a proof of Property C. To being with, we observe a simple, but useful observation.

Lemma 9.

$$h_j^{-1}x_i = N(\mathcal{X},\mathcal{H})\Delta_H^{-1}$$

where $N(\mathcal{X},\mathcal{H})$ represents a $\{\mathcal{X},\mathcal{H}\}$-negative word.

Proof. Since $z_H = z_G$ and $x_i = g_i\Delta_H^{-1}$, we have

$$z_H = x_i\Delta_H(g_i^{-1}z_Gg_1^{-1})x_1\Delta_H.$$

Therefore

$$h_j^{-1}x_i = (h_j^{-1}z_H\Delta_H^{-1})x_1^{-1}(z_G^{-1}g_1g_i)\Delta_H^{-1} = (h_j^{-1}h_1)x_1^{-1}(z_G^{-1}g_1g_i)\Delta_H^{-1}.$$

Since $z_G^{-1}g_i <_G 1$ and g_1 is $<_G$-minimal positive, $z_G^{-1}g_i \leq_G g_1^{-1}$, hence $z_G^{-1}g_1g_i \leq_G 1$. Thus, $(h_j^{-1}h_1)x_1^{-1}(z_G^{-1}g_1g_i)$ is written as an $\{\mathcal{X},\mathcal{H}\}$-negative word.

□

Now we are ready to prove Property C.

Proposition 3 (Property C). *Each non-trivial element $x \in X$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-positive word or an $\{\mathcal{X},\mathcal{H}\}$-negative word.*

Proof. Let x be a non-trivial element of X and take a reduced standard factorization of x,

$$\mathcal{F}(x) = rp_1q_1\cdots p_lq_l.$$

Recall that each p_i is written as an $\mathcal{X}$-positive word and each q_i is written as an $\mathcal{H}$-positive word. If $r \geq_H 1$, r is also written as an $\mathcal{H}$-positive or an empty word, hence we may express x as an $\{\mathcal{X},\mathcal{H}\}$-positive word.

By induction on $l = c(\mathcal{F})$, we prove that x is expressed by an $\{\mathcal{X},\mathcal{H}\}$-negative word under the assumption $r <_H 1$.

First assume that $q_1 \neq \Delta_H$. Since $r <_H 1$, we may express r as $r = N(\mathcal{H})h_1^{-1}$, where $N(\mathcal{H})$ is an $\mathcal{H}$-negative word. Take an $\mathcal{X}$-positive word expression of $p_1 = x_{i_1}x_{i_2}\cdots x_{i_p}$. Then by Lemma 9,

$$\begin{aligned} rp_1p_2q_2\cdots &= (N(\mathcal{H})h_1^{-1})(x_{i_1}x_{i_2}\cdots x_{i_p})q_1p_2q_2\cdots \\ &= N(\mathcal{H})(h_1^{-1}x_{i_1})x_{i_2}\cdots x_{i_p}q_1p_2q_2\cdots \\ &= N(\mathcal{X},\mathcal{H})\Delta_H^{-1}x_{i_2}\cdots x_{i_p}q_1p_2q_2\cdots \\ &= N(\mathcal{X},\mathcal{H})(h_1^{-1}x_{i_2})\cdots x_{i_p}q_1p_2q_2\cdots \\ &= \cdots \\ &= N(\mathcal{X},\mathcal{H})\Delta_H^{-1}q_1p_2q_2\cdots . \end{aligned}$$

$N(\mathcal{X},\mathcal{H})$ represents a $\{\mathcal{X},\mathcal{H}\}$-negative word.

Since S is a reduced standard factorization, $q_1 <_H z_H$. Since Δ_H is the $<_H$-maximal element of H which is strictly smaller than z_H, $Q_1 \leq_H \Delta_H$. We have assumed that $q_1 \neq \Delta_H$, the inequality is strict, so $(\Delta_H^{-1}q_1) <_H 1$. Therefore, we may write x as

$$x = N(\mathcal{X},\mathcal{H})(\Delta_H^{-1}q_1)p_2q_3\cdots p_{l-1}q_l.$$

Then $(\Delta_H^{-1}q_1)p_2q_3\cdots p_{l-1}q_l$ is a reduced standard factorization with complexity $(l-1)$. By induction, $(\Delta_H^{-1}q_1)p_2q_3\cdots p_{l-1}q_l$ is written as an $\{\mathcal{X},\mathcal{H}\}$-negative word, hence we conclude that x is written as an $\{\mathcal{X},\mathcal{H}\}$-negative word

Next assume that $q_1 = \Delta_H$. let $w = [g]$ be the maximal distinguished subfactorization of $\mathcal{F}(x)$ which contains q_1. Thus, the reduced standard factorization S is written as

$$\mathcal{F}(x) = rp_1'[g]x_up_s'q_sp_{s+1}\cdots p_lq_i$$

where $p_s' = x_u^{-1}p_s \in PX \cup \{1\}$.

Then by Lemma 9,

$$\begin{aligned} x &= rp_1'[g]x_up_s'q_sp_{s+1}\cdots p_lq_i \\ &= N(\mathcal{X},\mathcal{H})h_1^{-1}[g]x_u\Delta_H\Delta_H^{-1}p_s'q_s\cdots p_lq_l \\ &= N(\mathcal{X},\mathcal{H})h_1^{-1}[g]g_u\Delta_H^{-1}p_s'q_s\cdots p_lq_l \\ &= N(\mathcal{X},\mathcal{H})\Delta_H(z_G^{-1}gg_u)\Delta_H^{-1}p_s'q_s\cdots p_lq_l \end{aligned}$$

The distinguished subfactorization g is irreducible, hence $z_G^{-1}gg_u <_G 1$. This implies that $z_G^{-1}gg_u = \Delta_H^{-1}N(\mathcal{X},\mathcal{H})$, hence

$$\begin{aligned} &= N(\mathcal{X},\mathcal{H})\Delta_H\Delta_H^{-1}N(\mathcal{X},\mathcal{H})(\Delta_H^{-1}p_sq_s\cdots p_lq_l) \\ &= N(\mathcal{X},\mathcal{H})(\Delta_H^{-1}p_s'q_s\cdots p_lq_l) \end{aligned}$$

If $p_s' \neq 1$, then $(\Delta_H^{-1}p_s'q_s\cdots p_lq_l)$ is a reduced standard factorization. Hence by induction, $(\Delta_H^{-1}p_s'q_s\cdots p_lq_l)$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-negative word.

Since we have chosen the maximal distinguished sub factorization g, if $p_s' = 1$ then $q_s \neq \Delta_H$. Thus $q_s <_H \Delta_H$, and $(\Delta_H^{-1}q_s)p_{q+1}\cdots p_lq_l$ is a reduced standard factorization. By induction, $(\Delta_H^{-1}q_s)p_{q+1}\cdots p_lq_l$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-negative word.

Thus we conclude x is expressed by an $\{\mathcal{X},\mathcal{H}\}$-negative word. $\square$

2.6. **Proof of Theorem 1.** Now we are ready to prove main theorem.

Proof of Theorem 1. In Proposition 2 and Proposition 3, we have already confirmed the Property A and C for the generating set $\{\mathcal{X}, \mathcal{H}\}$. Hence by Corollary 1 the generating set $\{\mathcal{X}, \mathcal{H}\}$ indeed defines an isolated left ordering $<_X$ of X.

Now we show the ordering $<_X$ is independent of the choice of generating sets $\mathcal{G}$ and $\mathcal{H}$. Let $\mathcal{G}' = \{g'_1, \ldots\}$ and $\mathcal{H}' = \{h'_1, \ldots\}$ be other generating sets of G and H satisfying $[\mathbf{CF(G)}]$ and $[\mathbf{CF(H)}]$. Recall that $\Delta_H = z_H h_1^{-1}$ does not depend on a choice of a generating set $\mathcal{H}$. Let $x_i = g_i \Delta_H^{-1}$, $x'_i = g'_i \Delta_H^{-1}$, $\mathcal{X} = \{x_1, \ldots, \}$, and $\mathcal{X}' = \{x'_1, \ldots, \}$.

Since $\mathcal{H}$ and $\mathcal{H}'$ are generator of the same semigroup, we may write h_i as an $\mathcal{H}'$-positive word. Similarly, since $\mathcal{G}$ and $\mathcal{G}'$ are generator of the same semigroup, we may write g_i as a $\mathcal{G}'$-positive word $g_i = g'_{i_1} g'_{i_2} \cdots g'_{i_l}$. Thus,

$$x_i = g_i \Delta_H^{-1} = g'_{i_1} g'_{i_2} \cdots g'_{i_l} \Delta_H^{-1} = x'_{i_1} \Delta_H x'_{i_2} \Delta_H \cdots x'_{i_{l-1}} \Delta_H x'_{i_l}$$

so x_i is written as an $\{\mathcal{X}', \mathcal{H}'\}$-positive word. Thus, if $x \in X$ is expressed by an $\{\mathcal{X}, \mathcal{H}\}$-positive word, then x is also represented by an $\{\mathcal{X}', \mathcal{H}'\}$-positive word. By interchanging the role of $(\mathcal{G}, \mathcal{H})$ and $(\mathcal{G}', \mathcal{H}')$, we conclude that $\{\mathcal{X}, \mathcal{H}\}$ and $\{\mathcal{X}', \mathcal{H}'\}$ generates the same sub-semigroup of X. Hence they define the same isolated ordering of X.

(3) is obvious from the definition of $<_X$.

The inequality $h_1 <_X h_2 <_X \cdots <_X h_p$ is obvious from the definition. By Lemma 9, $x_i <_X h_1$ for all i. Now we show $x_i <_X x_j$ if $i < j$. Since $g_i <_G g_j$ if $i < j$, $g_i^{-1} g_j$ is written as a $\mathcal{G}$-positive word. Now by definition $g_i = x_i \Delta_H$, so we may express $\mathcal{G}$-positive word expression of $g_i^{-1} g_j$ as an $\{\mathcal{X}, \mathcal{H}\}$-positive word expression of the form $g_i^{-1} g_j = P(\mathcal{X}, \mathcal{H}) \Delta_H$. Therefore $x_i^{-1} x_j = \Delta_H g_i^{-1} g_j \Delta_H^{-1} = P(\mathcal{X}, \mathcal{H})$, so $x_i < x_j$. The assertion that $z = z_G = z_H$ is $<_X$-positive cofinal is obvious. To see $<_X$ is z-right invariant observe that $z^{-1} x_i z = x_i >_X 1$ and $z^{-1} h_j z >_X 1$. Now for $x, x' \in X$, assume $x <_X x'$, so $x^{-1} x'$ is written as $\{\mathcal{X}, \mathcal{H}\}$-positive word $w = s_1 \cdots s_m$, where s_i denotes x_j or h_j. Then $z^{-1}(x^{-1} x) z = (z^{-1} s_1 z) \cdots (z^{-1} s_m z) >_X 1$, hence $xz <_X x'z$. This completes the proof of (4).

To show (5), recall that by Lemma 3, we may choose the generating sets $\mathcal{G}$ and $\mathcal{H}$ so that the cardinal of $\mathcal{G}$, $\mathcal{H}$ are equal to $r(<_G)$, $r(<_H)$ respectively. Thus, $r(<_X) \leq r(<_G) + r(<_H)$.

Finally, we prove that $\langle x_1 \rangle$ is the unique $<_X$-convex non-trivial proper subgroup of X. Recall by (2), (4) and Lemma 1, x_1 is the minimal $<_X$-positive element of X, hence x_1 does not depend on a choice of $\mathcal{G}$ and $\mathcal{H}$. In particular, $\langle x_1 \rangle$ is a non-trivial $<_X$-convex subgroup.

Let C be a $<_X$-convex subgroup of X. Assume that $C \supset \langle x_1 \rangle$. Let $y \in C - \langle x_1 \rangle$ be an $<_X$-positive element. Then y is written as $y = x_1^m x_j P(\mathcal{X}, \mathcal{H})$ or $y = x_1^m h_l P(\mathcal{X}, \mathcal{H})$ where $m \geq 0$, $l > 0$, $j > 1$ and $P(\mathcal{X}, \mathcal{H})$ is an $\{\mathcal{X}, \mathcal{H}\}$-positive word. Since $x_1 \in C$, we may choose y so that $m = 0$.

First we consider the case $\mathcal{X} \not\subset \langle x_1 \rangle$. Then we may choose y so that $1 < x_2 \leq_X y$ holds, so the convexity assumption implies $x_2 \in C$. Now observe that $x_1^{-1} x_2 = \Delta_H g_1^{-1} g_2 \Delta_H^{-1} = \Delta_H P(\mathcal{X}, \mathcal{H})$, hence

$$1 <_X h_p \leq z_H h_1^{-1} = \Delta_H <_X \Delta_H P(\mathcal{X}, \mathcal{H}) = x_1^{-1} x_2$$

Since $x_1^{-1} x_2 \in C$, this implies $\mathcal{X} \cup \mathcal{H} = \{x_1, \ldots, x_k, h_1, \ldots, h_p\} \subset C$. Therefore we conclude $C = X$.

Next we consider the case $\mathcal{X} \subset \langle x_1 \rangle$. This happens only when $G = \mathbb{Z} = \langle g_1 \rangle$ and $z_G = g_1^N$. Then we may choose y so that $1 < h_1 \leq_X y$ holds, so $h_1 \in C$. Then $x_1^{-1}h_1 = \Delta_H g_1^{-1} h_1 = h_1^{-1} z_G g_1^{-1} h_1$ so $z_G g_1^{-1} = g_1^{N-1} \in C$. This implies $z_G = z_H \in C$, so $C = X$.

□

2.7. **Computational issues.** In this section we briefly mention the computational issue concerning the isolated ordering $<_X$. Let $G = \langle \mathcal{S} \,|\, \mathcal{R} \rangle$ be a group presentation and $<_G$ be a left ordering of G. The *order-decision problem* for $<_G$ is the algorithmic problem of deciding for an element $g \in G$ given as a word on $\mathcal{S} \cup \mathcal{S}^{-1}$ whether $1 <_G g$ holds or not. Clearly, the order-decision problem is harder than the word problem, since $1 <_G g$ implies $1 \neq g$. It is interesting to find an example of a left ordering $<_G$ of a group G, such that the order-decision problem for $<_G$ is unsolvable but the word problem for G is solvable.

There is another algorithmic problem which is related to the order-desicion problem of isolated orderings. We say a word on $\mathcal{G} \cup \mathcal{G}^{-1}$ is $\mathcal{G}$-*definite* if w is $\mathcal{G}$-positive or $\mathcal{G}$-negative, or empty. If $\mathcal{G}$ defines an isolated ordering of G, then every $g \in G$ admits a $\mathcal{G}$-definite word expression. *The $\mathcal{G}$-definite search problem* is a problem to find a $\mathcal{G}$-definite word expression of a given element of G.

Theorem 2. *Let us take $G, H, X, <_G, <_H, z_G, z_H, \mathcal{G}, \mathcal{H}, \mathcal{X}$ as in Theorem 1.*

(1) *The order-decision problem for $<_X$ is solvable if and only if the order-decision problem for $<_G$ and $<_H$ are solvable.*
(2) *The $\{\mathcal{X}, \mathcal{H}\}$-definite search problem is solvable if and only if the $\mathcal{G}$-definite search problem and the $\mathcal{H}$-search problem are solvable.*

Proof. Observe that since the restriction of $<_X$ to G and H yields the ordering $<_G$ and $<_H$ respectively, if the order-decision problem for $<_X$ is solvable, then so is for $<_G$ and $<_H$. Assume that $\{\mathcal{X}, \mathcal{H}\}$-definite search problem is solvable. It is easy to see that this implies $\mathcal{H}$-definite search problem is solvable. Since if $x \in G \subset X$, then $\{\mathcal{X}, \mathcal{H}\}$-definite word expression of x is naturally transformed into $\mathcal{G}$-positive word by using $g_i = x_i \Delta_H$ and $z_H = z_G$, hence $\mathcal{G}$-definite search problem is also solvable.

The proof of converse is implicit in the proof of Theorem 1 (1). Recall that in the proof of Property C (Proposition 3), we have shown that for a reduced standard factorization $\mathcal{F}(x) = rp_1q_1 \cdots p_l q_l$, $x >_X 1$ if $r >_H 1$ and $x <_X 1$ if $r_H < 1$. Moreover, the proof of Property C (Proposition 3) is constructive, hence we can algorithmically compute an $\{\mathcal{X}, \mathcal{H}\}$-negative word expression of x if $r <_H 1$ if the $\mathcal{G}$-definite search problem and the $\mathcal{H}$-search problem is solvable.

Thus, to solve the order-decision problem or $\{\mathcal{X}, \mathcal{H}\}$-definite search problem, it is sufficient to compute a standard factorization. We have established two different method to obtain a reduced standard factorization, in the proof of Lemma 6 and Lemma 8. Both proofs are constructive, hence we can algorithmically compute a reduced standard expression. □

It is not difficult to analyze a computational complexity of order-decision problem or the $\{\mathcal{X}, \mathcal{H}\}$-definite search problems based on the algorithm obtained from the proof of Proposition 3, Lemma 6 and Lemma 8. In particular, it is easy to observe the following result, whose proof is an easy exercise.

Proposition 4. *Let us take $G, H, X, <_G, <_H, z_G, z_H, \mathcal{G}, \mathcal{H}, \mathcal{X}$ as in Theorem 1.*

(1) *If the order-decision problem for $<_G$ and $<_H$ are solvable in polynomial time with respect to the input of word length, then the order-decision problem for $<_X$ is also solvable in polynomial time.*
(2) *If the $\mathcal{G}$-definite search problem and the $\mathcal{H}$-definite search problem is solvable in polynomial time, then $\{\mathcal{X},\mathcal{H}\}$-definite search problem is also solvable in polynomial time.*
(3) *Moreover, if one can always find a $\mathcal{G}$-definite and a $\mathcal{H}$-definite word expression whose length are polynomial with respect to the length of the input word, then one can always find $\{\mathcal{X},\mathcal{H}\}$-definite word expression whose length is polynomial with respect to the length of the input word.*

3. Examples, Observations and Conjectures

In this section we give examples of isolated left orderings produced by Theorem 1, and make simple observations and conjectures.

3.1. **Examples and new phenomenons.**

First we give examples of isolated left orderings produced by Theorem 1. All examples in this sections are new, and have various properties which previously known isolated orderings do not have.

Example 1. Let $a_1,\ldots,a_m$ $(m>1)$ be positive integers bigger than one and consider the group obtained as a central cyclic amalgamated free product of m infinite cyclic groups $\mathbb{Z}^{(i)}$ $(i=1,\ldots,n)$,

$$\begin{aligned} G = G_{a_1,\ldots,a_m} &= *_{\mathbb{Z}}\mathbb{Z}^{(i)} = \mathbb{Z}^{(1)} *_{\mathbb{Z}} (\mathbb{Z}^{(2)} *_{\mathbb{Z}} (\cdots (\mathbb{Z}^{(m-1)} *_{\mathbb{Z}} \mathbb{Z}^{(m)})\cdots)\\ &= \langle x_1,\ldots,x_m \mid x_1^{a_1} = x_2^{a_2} = \cdots = x_m^{a_m}\rangle \end{aligned}$$

By Theorem 1, the group G has an isolated left ordering $<_G$.

The group G is the simplest example of groups and isolated orderings constructed by Theorem 1, but nevertheless has various interesting properties which have not appeared in the previous examples:

(1): $r(<_G)=m$.

Since G is an amalgamated free product of m infinite cyclic groups, the rank of G is m. On the other hand, Theorem 1 (5) says $c(<_G)\leq m$. Hence the rank of isolated ordering $<_G$ is m.

(2): The isolated orderings $<_G$ of G is not derived from Dehornoy-like orderings if $m>2$.

As we mentioned, the special kind of left-orderings called *Dehornoy-like orderings* produces isolated orderings, and all previously known examples of genuine isolated orderings are derived from Dehornoy-like orderings.

In [3] it is proved that an isolated ordering derived from Dehornoy-like orderings has a lot of convex subgroups: if the isolated orderings $<_H$ of a group H is derived from the Dehornoy-like orderings, then there is at least $r(<_H)-1$ proper, $<_H$-convex nontrivial subgroups.

On the other hand Theorem 1 (6) shows the isolated orderings $<_G$ has only one proper, $<_G$-convex nontrivial subgroups. This implies that the isolated orderings $<_G$ of G is not derived from Dehornoy-like orderings if $m>2$. This provides a

counter example of somewhat optimistic conjecture: every genuine isolated ordering is derived from Dehornoy-like ordering. We remark that in the case $m = 2$, the ordering is the same as the ordering constructed in [3], hence it is derived from Dehornoy-like ordering.

(3): The natural right G-action on $LO(G)$ has at least 2^m distinct orbits derived from isolated orderings.

There is a natural, continuous right G-action on $\mathrm{LO}(G)$, defined as follows: For a left ordering $<$ of G and $g \in G$, we define the left ordering $< g$ by $h\,(<\cdot g)\,h'$ if $hg < h'g$. This action sends an isolated ordering to an isolated ordering. Little is known about the quotient $\mathrm{LO}(G)/G$.

Observe that we have two choices of isolated orderings for each infinite cyclic group factor $\mathbb{Z}^{(i)}$. Thus by Theorem 1 we can construct 2^m distinct isolated left orderings of G. It is easy to see all of them belong to distinct G-orbits. Hence, we have at least 2^m different G-orbits derived from isolated orderings. Recall that $m = r(<_G)$ is equal to $r(G)$, the rank of G.

(4): The natural right $Aut(G)$-action on $LO(G)$ has at least 2^m distinct orbits derived from isolated orderings if all $a_1, \ldots, a_m$ are distinct.

As in the group G itself, there is a natural right $\mathrm{Aut}(G)$-action on $\mathrm{LO}(G)$. For a left ordering $<$ of G and $\theta \in \mathrm{Aut}(G)$, we define the left ordering $<\cdot\theta$ by $h <\cdot\theta\ g$ if $(h)\theta < (g)\theta$. The right G-action on $\mathrm{LO}(G)$ can be regarded as the restriction of the natural $\mathrm{Aut}(G)$-action to the subgroup $\mathrm{Inn}(G)$. As in (3), if all $a_1, \ldots, a_m$ are distinct, then we have 2^m distinct $\mathrm{Aut}(G)$-orbit derived from isolated orderings.

Example 2. Next we consider the construction of the case z_H is non-central. First of all, let $G_{m,n} = \langle b, c \,|\, b^m = c^n \rangle$. By Example 1, $G_{m,n}$ has an isolated left ordering $<_{m,n}$ which is defined by $\{bc^{1-n}, c\}$.

Then $bc^{1-n} \cdot b^m = bc$ is non-central element, but is $<_{m,n}$-positive cofinal. $<_{m,n}$ is (bc^{1-n})-right invariant by Lemma 1, and $<_{m,n}$ is also b^m-right invariant, since b^m is central. Thus, $<_{m,n}$ is (bc)-right invariant.

Thus, we can take bc as an element z_H in Theorem 1. Now we consider the group $G'_{p,q,m,n} = \mathbb{Z} *_{\mathbb{Z}} G_{m,n} = \mathbb{Z} *_{\mathbb{Z}} (\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z})$ defined by

$$\langle a, b, c \,|\, b^m = c^n, a^p = (bc)^q \rangle$$

This group has an isolated left ordering $<_G$, defined by $\{a(bc)^{1-q}, bc^{1-n}, c\}$ Let us put $x = a(bc)^{1-q}$, $y = (bc)^{1-n}$, and $z = c$. Then the group $G_{p,q,m,n}$ is presented as

$$G'_{p,q,m,n} = \langle x, y, z \mid (yz^{n-1})^m = z^n, (x(yz^n)^{q-1})^p = (yz^n)^q \rangle$$

Now we observe a remarkable feature of the group $G'_{p,q,m,n}$.

(5): The center of $G' = G'_{p,q,m,n}$ is trivial.

Since G' is an amalgamated free product of $G_{m,n}$ and $\mathbb{Z}$, central element in G' is written as $a^{pN} = (bc)^{qN}$ for some N. However, if $N \neq 0$, $(bc)^{qN}$ do not commute with b, hence it is not central.

We remark that all previously known examples of groups having isolated left ordering has non-trivial (in fact, infinite cyclic) center. This is the first example

of group having isolated left ordering with trivial center. In a similar manner, we can construct many new examples of groups having isolated ordering with trivial center.

3.2. **Some conjectures.** Based on the above examples and observations, we here give some conjectures which is related to the structure of $\mathrm{LO}(G)$ and groups having isolated orderings.

First of all, it is suspected that under the natural right G-action on $\mathrm{LO}(G)$, the number of orbits derived from isolated orderings is finite. It seems to be reasonable to pose the following stronger conjecture:

Conjecture 1. For a finitely generated group G, the number of G-orbits derived from isolated ordering is at most $2^{r(G)}$, where $r(G)$ is the rank of G. The upper bound is achieved only if $G = G_{a_1,\ldots,a_m}$ given in Example 1.

Second, observe that all known examples of groups having isolated ordering has non-trivial abelian quotient. The partially central cyclic amalgamation preserves the property that having non-trivial abelian quotient. This leads the following conjecture:

Conjecture 2. If a group G has an isolated left ordering, then $H_1(G;\mathbb{Z})$ is non-trivial. More strongly, $H_1(G;\mathbb{Q})$ is non-trivial.

Though Conjecture 2 is stated in algebraic form, Conjecture 2 concerns the dynamics of the commutator subgroup $[G,G]$. Recall that for a countable group G, a left ordering of G induces an orientation-preserving faithful action of G on the real line $\mathbb{R}$ [5]. The meaning of Conjecture 2 in dynamics point of view is, for the action of G induced from isolated left ordering, the dynamics of G and $[G,G]$ have different features, which implies in particular $G \neq [G,G]$.

In this direction, we pose another conjecture. A group G is said to have (Serre's) *Property FA* if every action of G on a tree has a global fixed point. If a group has Property FA, then G is not written as an amalgamated free product or as an HNN extension [8]. Thus, all groups constructed from the partially central cyclic amalgamation construction do not have Property FA. The Dehornoy-like orderings also seems to be related to certain action on tree. Indeed, many property of Dehornoy-like orderings in [3] were obtained by studying an action on tree. Thus, it seems to be reasonable to expect that isolated orderings are related to the group action on tree. In particular, we make the following conjecture.

Conjecture 3. If a finitely generated group G has an isolated left ordering, then G does not have Property FA.

Conjecture 3 is also related to Conjecture 2, since a finitely generated group G having Property FA satisfies $H_1(G;\mathbb{Q}) = 0$. Thus, affirmative answer for stronger version of Conjecture 2 implies the affirmative answer of Conjecture 3.

The Serre's Property FA is also related to other property of groups, such as Kazhdan's property (T). It is also interesting problem to study how these properties of groups are related to existence of isolated orderings.

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